TRANSITIONING FROM WHOLE NUMBERS TO INTEGERS

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This paper presents the results of a pre-test, instruction, post-test study that investigated students' integer mental models and how their models changed based on instruction. Sixty-one first graders' responses to questions about the values and order of negative numbers were categorized according to a series of mental models. The models reveal that initially, students over-rely on the values and/or order of whole numbers to varying degrees when working with negative numbers. Focused exploration on the properties of negative numbers helped students transition to more sophisticated mental models compared to only learning about moving in positive and negative directions on number lines.

Keywords: Number Concepts and Operations; Elementary School Education

Purposes of the Study

During students' progression through early elementary school, one of the foundational mathematical understandings they must develop is that of numbers (National Research Council, 2001; National Council of Teachers of Mathematics, 2000). During this time, students learn the counting sequence, numeral names, and numerical values. In particular, students must coordinate their knowledge of number values and order—understanding that moving one number forward in the counting sequence corresponds to a one-unit increase in value—so that they can use this information to solve addition and subtraction problems (Griffin, Case, & Capodilupo, 1995; Fuson, 2004). As students move along the continuum of numerical understanding, difficulty arises when they must expand their number knowledge to include negative numbers and rethink how numerals relate to numerical values: 10 is big in terms of positive numbers but small when made negative; the inclusion of negative numbers also requires students to reevaluate the meanings of addition and subtraction (e.g., adding no longer always results in a larger number) (Bruno & Martinon, 1999; Küchemann, 1980; Murray, 1985). Further, students must navigate the changing meanings of the minus sign (it can mean an operation, a negative sign, or an indication to take the opposite), which later interferes with students' ability to reduce polynomials and manipulate algebra problems (Gallardo & Rojano, 1994; Vlassis, 2004, 2008).

Past research on negative number instruction and student learning explored whether students could learn how to solve integer addition and subtraction problems from a particular method of instruction (Linchevski & Williams, 1999; Schwarz, Kohn, & Resnick, 1993; Thompson & Dreyfus, 1988) or whether one method of instruction is more effective than another (Liebeck, 1990; Janvier, 1985). Results from these and other studies highlight strategies students use to solve integer problems. For example, to solve –4 + –3, a student might add 4 + 3 = 7 and then add a negative sign to get –7 (Bofferding, 2010). Another student might treat negatives as worth zero and get an answer of –4 because adding zero will not change the answer (Bofferding, 2011).

What might account for these differences? One influence on students' solutions to integer addition and subtraction problems is how they think about the values and order of negative numbers in comparison to positive numbers. However, research does not provide a clear picture of what the transition from whole number to integer understanding looks like. This study contributes to this area by addressing the following research questions: (1) What are first grade students' mental models of negative integers in relation to order and value (the elements underlying the central conceptual structure of numbers)? (2) How do students' mental models change based on integer instruction?

Theoretical Framework

The *central conceptual structure of number* (CCSN) or *mental number line* is a mental model or internal structure hypothesized to support numerical thinking (Case, 1996; Griffin, Case, & Capodilupo, 1995). This structure involves four components: number word order (i.e., the counting sequence), a tagging routine for counting objects, numerical values, and written symbols. Although young children may learn how to say the number names in order, they do not initially use this process to quantify sets; similarly, they might compare two sets of objects visually instead of counting and comparing them numerically (Sophian, 1987). Eventually children coordinate the four components, creating the fully integrated CCSN. By referring to this mental model and counting up and down their mental number line, students can add and subtract single-digit numbers (Case, 1996; Griffin, Case, & Capodilupo, 1995).

As students learn about negative numbers, they need to modify their CCSN. Because there are no true physical manifestations of negative numbers, students need to reason that numbers further to the left on the mental number line are smaller than numbers to their right, even if before zero and even if they contain the same numerals. Therefore, while -2 and 2 look similar and are equally far away from zero, the number further to right on the number line is greater. Students must also wrestle with the changing meaning of the minus sign; for example, –4 – –3 does not mean subtract 3 twice (Murray, 1985) but take away –3 from –4.

Situations, such as this one, where students must reorganize their knowledge structures, involve conceptual change. In the context of numbers, students have *initial* mental models that numbers are discrete, which arise from their experiences with objects (Gallistel & Gelman, 1992; Vamvakoussi & Vosniadou, 2004). Students who have an initial mental model for negative numbers might ignore the negative signs and treat the numbers as if they are positive; for example, they might place negative numbers next to their positive counterparts when ordering them (Peled, Mukhopadhyay, & Resnick, 1989). As children have more experiences with a concept, they begin to restructure their initial mental models in order to deal with new, conflicting information; this process results in one of many *synthetic* or *intermediary* mental models. For example, when students learn that negatives exist, they might think that –7 is greater than –3 because 7 is greater than 3 (Stavy, Tsamir, & Tirosh, 2002). With time and experience students will eventually develop the *formal* mental model, where negatives are ordered opposite their positive counterparts on the number line—with zero separating positive and negative values—and numbers decreasing in value to the left and increasing in value to the right. The goal of this study is to address the research questions by using the lens of conceptual change to investigate how students' mental models for the CCSN change during the transition from whole number to integer understanding.

Methods

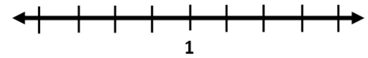
Participants and Site

This study took place at the end of the school year in a diverse elementary school in California. It was important to recruit students who had initial mental models for negative numbers so that I could explore the transitions they make as they learn about the topic. Based on pilot work, I selected and recruited students at the end of their first grade year. Overall, 61 first graders participated.

Materials and Data Collection

Interviews. The study consisted of a pre-test, instructional intervention, and a post-test. Both pre- and post-tests were conducted as individual interviews by a trained graduate student and me. Additionally, the two tests had the same questions, although with different numbers. The goal of the questions was to determine students' understanding of negative number values, order, and symbols (see Table 1). Although we also asked students arithmetic questions, results for those data are not discussed here. After students solved a question, we asked them to explain how they solved it, and students did not receive feedback on whether they answered correctly.

Table 1: All Pre-Test and Post-Test Questions for Each Category

Category	Number	Pre-Test Questions	Post-Test Questions
Counting	1	*Start at five and count backwards as far as you can. Can you count back any further? Is there anything less than <last number said>?*	
Number Line	1	*Fill in the missing numbers on the number line.* 	
Ordering	2	*Put these number cards in order from the least to the greatest. Which is the least? Which is the greatest?*	
		a. $\boxed{2}$, $\boxed{-3}$, $\boxed{0}$, -$\boxed{9}$, $\boxed{3}$, $\boxed{8}$, $\boxed{-5}$	a. $\boxed{5}$, $\boxed{-2}$, $\boxed{-8}$, $\boxed{1}$, $\boxed{7}$, $\boxed{-4}$
		b. $\boxed{6}$, $\boxed{-6}$, $\boxed{4}$, $\boxed{-7}$, $\boxed{3}$, $\boxed{-1}$	b. $\boxed{3}$, $\boxed{8}$, $\boxed{-9}$, $\boxed{-7}$, $\boxed{0}$, -$\boxed{5}$
Greater	7	*What are these two numbers? Circle the one that is greater.*	
		a. 8 vs. 6	a. 6 vs. 4
		b. 3 vs. –9	b. 5 vs. –7
		c. –2 vs. –7	c. –3 vs. –1
		d. –5 vs. 3	d. –8 vs. 4
		e. –8 vs. –2	e. –6 vs. –2
		Two children are playing a game and trying to get the highest score. Circle who is winning.	
		f. Abigail: 4 vs. Joseph: –7	f. Amy: 5 vs. Ken: –9
		g. Crystal: –7 vs. Leon: –3	g. Dan: –8 vs. Will: –6

Instructional intervention. Based on their performance on the pre-test, students were stratified and randomly assigned to one of three instructional groups. The groups were designed to provide students with differing levels of exposure to negative numbers so that I could explore the ways in which their mental models change. I taught each instructional group for eight, 45-minute sessions and followed strict lesson plans to maintain proper instructional treatments.

The *Integer Operations* group only had exposure to negatives. They learned how to use a number line to move *more positive, more negative, less positive, and less negative,* but they did not learn specifics about negative numbers. The *Integer Properties* group learned about the values and order of negative numbers as well as how to tell the difference between positive and negative numbers and minus and negative signs. Finally, the *Combined Integer Instruction* group had three lessons in common with the *Integer Properties* group and five in common with the *Integer Operations* group. Therefore, they learned the order and value of negatives as well how to use movements of more and less on a number line to add and subtract integers.

Data Analysis

To analyze the data, I first transcribed students' responses to the pre- and post-test questions and classified them as correct or incorrect. The counting and number line tasks needed to include negative numbers to be considered correct. Based on the methods employed by Vosniadou and Brewer (1992) to identify mental models of the Earth, I first formulated possible integer mental models using previous results from the literature (Peled et al., 1989) and the CCSN as a guide. For each test, I then coded students according to the mental model that would account for the pattern of their responses to the interview questions. As I coded, I created additional mental model categories when necessary to capture differences in students' responses that were not depicted by the initial mental models hypothesized from the literature. As found by Vosniadou and Brewer, some students fell into categories that were mixed versions of two other categories, which suggests they have inconsistent mental models. Once all students were categorized according to an integer mental model, I compared the percentage of students with each mental model in each instructional group before and after instruction.

Results

Overall, across the pre-test and post-test, students demonstrated a variety of mental models for integers, reflecting initial, intermediary, or formal understanding of the concept.

Initial Mental Models (in ascending order)

Whole number mental model. Students with a whole number mental model treated all negatives as if they were positive. When counting backward they stopped at zero or one, their number lines only included whole numbers, they ordered negative numbers next to positive numbers (e.g., 0, 3, –5, –7, 8, –9) and choose greater integers based on absolute value (e.g., –9 > 3).

Continuous zero mental model. Although students with a continuous zero mental model also treated negatives as if they were positive, these students used repeating zeroes on their number lines and/or when counting backward as demonstrated by Student 403: "Five, four, three, two, one, zero, zero, zero." Similarly, Bishop, Lamb, Philipp, Schappelle, and Whitacre (2011) had three first graders label multiple zeroes when playing a number line game. Their treatment of zero suggests that these students have some idea that the mental number line continues indefinitely less than one.

Absolute value mental model. The absolute value mental model is the first which involves students noticing a difference between positive and negative numbers. Students with this mental model separated the negative numbers from the positive numbers when ordering them (sometimes correctly) but continued to claim that negative numbers have the same values as their positive counterparts. Therefore, while they might have correctly ordered negatives before zero, they still claimed that –9 is greater than 3 and –7 is greater than –2.

Symbolic mental model. Students with the symbolic mental model correctly counted into the negatives. Additionally, when filling in the number line, they often did so correctly but used their own notation to indicate negative numbers. For example, one student wrote "N3" for negative three, while another student used an "X" instead of "N." Consequently, when ordering and determining which integer was greater, they treated all negative numbers as if they were whole numbers because the problem notation did not match their invented notation. This result highlights the importance of the role of symbols in the central conceptual structure of integers. It is possible they could have ordered the numbers using their own notation, but, unfortunately, this was not tested. Therefore, the students' understanding in this category may be understated.

Ordered nothings mental model. A few students not only separated negative numbers from positive numbers when ordering them but also treated negatives as worth zero. One student justified her order of the numbers (–3, –5, –9, 0, 2, 3, 8) explaining that "nine minus nine is zero," as is five minus five and three minus three. Later, when comparing –2 and –7 she explained that although 7 is greater than 2, both of these numbers were zero.

Intermediary Mental Models (in ascending order)

Separated value mental model. The separated value model is the first example of a mental model where students start to add to their CCSN instead of trying to fit negative numbers into positive number rules. Students with this mental model could correctly order integers and could determine the larger of two negative integers *or* the larger of two positive integers. However, when given a group of positive and negative integers, the students ignored the negative numbers when determining which number in the group was greatest or least. Therefore, a student in this category might say that –3 is greater than –5 and 2 is greater than –3 but when given –5, –3, 2, and 8 would say that 2 is the least. This behavior may arise from students thinking that anything less than zero is not a real quantity.

Equal/Unequal magnitude (mixed) mental model. Students with this mixed model gave responses consistent with having a whole number mental model for some order and value questions and a magnitude mental model for others.

Magnitude mental model. Students with a magnitude mental model ordered negatives before zero but either reversed their order (e.g., –1, –6, –7, 3, 4, 6) and claimed that –7 is greater than –2 or ordered the negatives correctly but thought that numbers further away from zero were larger, again claiming that –7 is greater than –2.

Equal/Unequal integer (mixed) mental model. Several students treated negatives as positive while ordering them but then correctly determined the greater of two integers for all combinations of integer pairs. These students had both whole number and integer mental models.

Dual value (mixed) mental model. Students with the dual value mixed mental model always identified negatives as smaller than positive numbers but sometimes treated negatives further away from zero (e.g., –9) as larger and sometimes correctly identified them as smaller than negative numbers closer to zero (e.g., –1). It is possible that students with these mixed mental models were in the process of transitioning from relying on one mental model to the other. On the other hand, their responses could have been influenced by the context of the question.

Formal Integer Mental Model

Students with a formal integer mental model correctly ordered negative numbers in relation to positive numbers and consistently identified the greater integer regardless of whether two or several integers were presented.

Changes in Integer Mental Models

While students across instructional groups had a similar spread of mental models on the pre-test (15–17 students in each group started with initial mental models), there were several changes by the post-test. Table 2 shows how students' mental models shifted in each instructional group. The cells show the percentages of students in each instructional group who started with a particular integer mental model (initial, intermediary, or formal) on the pre-test and ended with a particular mental model on the post-test. Students who started with initial mental models and shifted to formal mental models made the greatest transition.

As shown in Table 2, most students in the Integer Operations group (who did not learn about the properties of negatives) started with initial mental models on the pre-test and still had initial mental models on the post-test. On the contrary, most students in the other two groups transitioned from having initial mental models on the pre-test to having intermediary or formal mental models of integers on the post-test.

Table 2: Percentage of Students with each Mental Model on Pretest and Post-test by Instructional Group

		Mental Models on (Pretest, Post-Test)					
Group	***n***	**(I, I)**	**(M, M)**	**(F, F)**	**(I, M)**	**(M, F)**	**(I, F)**
Combined Instruction	20	20%	0%	10%	**35%**	5%	**30%**
Integer Properties	20	5%	5%	10%	**35%**	10%	**35%**
Integer Operations	21	**62%**	5%	14%	10%	0%	10%

Note. I = Initial Mental Model; M = Intermediary Mental Model; F = Formal Mental Model. A student who falls in the (I, I) category demonstrated an initial mental model on both tests.

To test whether the average differences in mental model advancement were due to the instructional treatments versus chance, I used the Kruskal-Wallis 1-way ANOVA by ranks (Shavelson, 1996). Students who had reached ceiling (i.e., had formal mental models) on the pre-test were eliminated from the analysis, which resulted in 18 people per instructional group. Results indicate a significant effect for instructional

Van Zoest, L. R., Lo, J.-J., & Kratky, J. L. (Eds.). (2012). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Kalamazoo, MI: Western Michigan University.

group ($H_{observed}$=9.8064, $H_{critical}$=5.99, α<.05). Pairwise comparisons reveal that the Combined Instruction (mean rank = 31) and Integer Properties (mean rank = 34) groups improved significantly in terms of developing more formal mental models for negative numbers than the Integer Operations group (mean rank = 17) (*HSD* = 7.76, α < .05). There was no significant difference between the Combined Instruction and Integer Properties groups.

Students in the Combined Instruction and Integer Properties groups both had instruction on the properties of negative numbers; however, only one student from the Integer Properties group continued to have an initial mental model for integers at the post-test. Furthermore, this student progressed from treating all negatives as positive (Whole Number Mental Model) to interpreting the value of negative numbers as different from positive numbers (Ordered Nothing Mental Model). On the other hand, four students in the Combined Instruction group continued to have initial mental models for integers at the post-test, and all four treated negative numbers as if they were positive (Absolute Value and Continuous Zero Mental Models). This difference is suggestive that spending focused time (more than 3 days) on integer properties helped students in the Integer Properties group develop more advanced mental models. This result would need to be studied further, though, to determine how generalizable it is and whether it can be replicated.

Discussion and Implications

The results of this study highlight mental models that students develop for integers and provide insight into the process through which students, who have developed the central conceptual structure of number, expand or restructure their conceptual structure to include integers. As found in other research on conceptual change (Vosniadou & Brewer, 1992), the students' initial mental models for integers were constrained based on their current knowledge (in this case, their knowledge of whole numbers). Their understanding of how order, values, and numerical symbols relate led students to interpret negative numbers as a different type of positive number. Some students with initial mental models ordered negatives apart from positive numbers but did not use this information to reason that the values would also be different; meanwhile, others sorted them separately and considered negatives as amounts taken away, equivalent to zero. Further, students with intermediary mental models knew that negatives were less than zero but had not associated them with the order of numbers definitively.

Identifying students' integer mental models can help teachers better understand students' incorrect solutions to integer arithmetic problems. Further, the results of this study have implications for how curricula and instruction could support students' learning of abstract concepts and help move them along the continuum of numerical understanding. Based on the integer mental models identified, several concepts to emphasize in integer instruction over several lessons include distinguishing between the negative sign and the minus sign, understanding the symmetrical nature of the number line, distinguishing between greater distances from zero versus greater numerical values, and identifying the difference between negative numbers and zero. As with other notation and representations, students need formal explorations with negatives because exposure to them does not make their structure obvious. Students in the Integer Operations group did not learn the structure and values of negative numbers through hearing the words "positive" and "negative" without associating the words with their symbols and meaning. Instruction on negatives (and other abstract ideas) needs to help students see the structure of the concept: there is symmetry in the number sequence when zero separates the positive from the negative numbers, numbers increase as we count up through the sequence, and the minus sign takes on new roles (Vlassis, 2008). Furthermore, teachers can use familiar representations to extend students' thinking; classroom number lines or number paths should continue beyond zero into negatives (even in the younger grades), so that as students learn about negative numbers, they receive constant visual feedback of their existence.

Although curriculum developers, researchers, and policymakers continue to place negative numbers late in the curriculum—6th and 7th grades (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010)—this study demonstrates that students are quite capable of learning about integers much earlier than fifth grade. In fact, students in this study were able to learn about

negatives as early as the end of first grade, although whether this would be beneficial in the long term remains to be investigated.

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